The regularized 3D Boussinesq equations with fractional Laplacian and no diffusion

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Abstract

In this paper, we study the 3D regularized Boussinesq equations. The velocity equation is regularized à la Leray through a smoothing kernel of order α in the nonlinear term and a β-fractional Laplacian; we consider the critical case $\alpha + \beta = \frac{5}{4}$ and we assume $\frac{1}{2} < \beta < \frac{5}{4}$. The temperature equation is a pure transport equation, where the transport velocity is regularized through the same smoothing kernel of order α. We prove global well posedness when the initial velocity is in H^r and the initial temperature is in $H^{r-\beta}$ for $r > \max(2\beta, \beta + 1)$. This regularity is enough to prove uniqueness of solutions. We also prove a continuous dependence of the solutions on the initial conditions.

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1. Introduction

We consider the Boussinesq system in a d-dimensional space:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p = \theta e_d \\ \partial_t \theta + v \cdot \nabla \theta = 0 \\ \nabla \cdot v = 0 \end{cases} \tag{1}$$

where $v = v(t, x)$ denotes the velocity vector field, $p = p(t, x)$ the scalar pressure and $\theta = \theta(t, x)$ a scalar quantity, which can represent either the temperature

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of the fluid or the concentration of a chemical component; e_d is the unit vector $(0, \ldots, 0, 1)$, the viscosity ν is a positive constant. Suitable initial conditions v_0, θ_0 and boundary conditions (if needed) are given.

For $d = 2$, the well posedness of system (1) in the plane has been studied by several authors under different assumptions on the initial data (see [12, 7, 1, 11, 8, 9]). For $d = 3$, very little is known; it has been proven that there exists a local smooth solution. Some regularity criterions to get a global (in time) solution have been obtained in [21, 10]. Otherwise, in the particular case of axisymmetric initial data, [2] shows the global well posedness for the Boussinesq system in the whole space.

To overcome the difficulties of the three-dimensional case, different models have been proposed. For instance, one can regularize the equation for the velocity by putting a fractional power of the Laplacian; this hyper-dissipative Boussinesq system takes the form

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nu(-\Delta)^\beta v + \nabla p = \theta e_3 \\ \partial_t \theta + v \cdot \nabla \theta = 0 \\ \nabla \cdot v = 0 \end{cases} \tag{2}$$

For $\beta > \frac{5}{4}$, [28] proved the global well posedness. This result has been improved by Ye [27], allowing $\beta = \frac{5}{4}$.

Notice that for zero initial temperature θ_0, the Boussinesq system reduces to the Navier-Stokes equations. It is well known that the three-dimensional Navier-Stokes equations have either a unique local smooth solution or a global weak solution. The questions related to the local smooth solution being global or the global weak solution being unique are very challenging problems that are still open since the seminal work of Leray. For this reason, modifications of different types have been considered for the three-dimensional Navier-Stokes equations. On one side there is the hyper-viscous model, i.e. (2) with zero initial temperature; when $\beta \geq \frac{5}{4}$, uniqueness of the weak solutions has been proved in [17] (see Remark 6.11 of Chapter 1) and [18] . On the other hand, Olson and Titi in [20] suggested to regularize the equations by modifying two terms. For a particular model of fluid dynamics, they replaced the dissipative term by a fractional power of the Laplacian and they regularized the bilinear term of vorticity stretching à la Leray. The well posedness of those equations is obtained by asking a balance between the modification of the nonlinearity and of the viscous dissipation; at least one of them has to be strong enough, while the other might be weak. Similarly, Barbato, Morandin and Romito in [4] considered the Leray-α Navier-Stokes equations with fractional dissipation

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \nu(-\Delta)^\beta v + \nabla p = 0 \\ v = u + (-\Delta)^\alpha u \\ \nabla \cdot u = \nabla \cdot v = 0 \end{cases} \tag{3}$$

and proved that this system is well posed when $\alpha + \beta \geq \frac{5}{4}$ (with $\alpha, \beta \geq 0$); even some logarithmic corrections can be included, but we do not specify this detail,

since it is not related to our analysis. It is worth mentioning the result of the current authors with Barbato in [3], where a stochastic version of the associated inviscid system to (3) (when $\nu = 0$) has been studied. In fact, by choosing an appropriate stochastic perturbation of the system to be formally conservative, they were able to prove global existence and uniqueness of solutions in law for $\alpha > \frac{3}{4}$. This is a very strong result although the uniqueness has to be understood in law.
Similar regularization have been used for the MHD models, see e. g. [26] and the references therein. Since these models are quite different from the ones considered in the current paper, we don't state their results and we refer interested readers to the literature related to these models. The physical motivation of these regularization defined in terms of smoothing kernels is related to a subgrid length scale in the model and these kernels work as a kind of filter with certain widths. An extensive explanation of these models can be found in [20] and the references therein.

Inspired by [4], in this paper we consider the modified Boussinesq system for $d = 3$, where the equation for the velocity has fractional dissipation whereas the temperature equation has no dissipation term; a Leray-regularization for the velocity appears in the quadratic terms. This system is

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \nu(-\Delta)^\beta v + \nabla p = \theta e_3 \\ \partial_t \theta + u \cdot \nabla \theta = 0 \\ v = u + (-\Delta)^\alpha u \\ \nabla \cdot u = \nabla \cdot v = 0 \end{cases} \tag{4}$$

As in [20], we work on a box and assume periodic boundary conditions.

Inspired by Ye [27] and Barbato, Morandin, Romito [4], our goal is to prove well posedness of system (4) for $\alpha + \beta = \frac{5}{4}$ when v_0, θ_0 are regular enough. So the interesting case is for $\beta < \frac{5}{4}$; indeed, the result of Ye corresponds to $\alpha = 0$ and $\beta \geq \frac{5}{4}$ and that of Barbato, Morandin, Romito does not include the temperature equation, i.e. corresponds to our system (4) with $\theta_0 = 0$. We have to point out that the temperature satisfies a pure transport equation, without thermal diffusivity; hence, the uniqueness result for the unknown θ requires the velocity to be smooth enough and this imposes β to be not too small. We point out that [15] and [22] deal with some regularized Boussinesq system similar to (4); however they consider an easier case, since they set $\alpha = \beta = 1$, the bilinear term in the equation for the velocity is $(u \cdot \nabla) u$ instead of our $(u \cdot \nabla) v$ and the equation for the temperature is dissipative, i.e. there is the term $-\kappa \Delta \theta$ in the l.h.s..

We can summarize our result in the following

Theorem 1. *Assume* $\frac{1}{2} < \beta < \frac{5}{4}$ *with*

$$\alpha + \beta = \frac{5}{4}.$$

Then, system (4) has a unique global smooth solution for any smooth initial conditions v_0, θ_0*.*

Remark 1. Notice that assuming $\alpha + \beta = \frac{5}{4}$, the condition $\frac{1}{2} < \beta < \frac{5}{4}$ is equivalent to $0 < \alpha < \frac{3}{4}$. Let us notice that our techinque works also in the easier case $\alpha + \beta > \frac{5}{4}$ with $\beta > \frac{1}{2}$, $\alpha \geq 0$ (see Remark 5); but the result of [27] for $\alpha = 0$ and $\beta = \frac{5}{4}$ cannot be obtained with our techinque.

Our proofs rely on the commutator estimates introduced in [14], also used in [27]. However in contrast to [27], we first prove global existence (for any $\alpha \geq 0$ and $\beta > 0$) and then uniqueness of these solutions; moreover we consider different order of space regularity for v and θ (H^r-regularity for v and $H^{r-\beta}$-regularity for θ), whereas in [27] the same order of regularity for both v and θ is considered. We point out that the requirement on the regularity on the initial data is needed only to guarantee uniqueness.

The paper is organized as follows. Section 2 is devoted to the mathematical framework. Our main functional spaces, the regularization operator Λ^s with its properties given in Lemma 5 are defined. The bilinear operator of the Navier-Stokes equations, the transport operator and the commutator operator are defined and their properties are stated in Lemma 2, Remark 2 and Remark 3 and Lemma 4. The main system is then written in its abstract (operator) form and the definition of weak solutions is given. At the end of this section, we recall the Gagliardo-Nirenberg and Brézis-Gallouet-Wainger inequalities and some continuity results. In Section 3, we prove global existence of weak solutions with their uniform estimates. Slightly better estimates are performed. However, they are not enough to prove the uniqueness of solutions. The main result of the paper is stated in Section 4, Theorem 10, where we prove global existence of regular solutions; this regularity is enough to prove uniqueness of solutions and their continuous dependence with respect to the initial conditions, see Theorem 11 and Theorem 12. Let us point out that the results of Section 4 provide Theorem 1, i.e. every smooth initial data gives rise to a unique smooth solution. Section 5 is devoted to showing in more details the crucial estimates used in Section 4.

2. Mathematical framework

We consider the evolution for positive times and the spatial variable belongs to a bounded domain of $\mathbb{R}^3$; for simplicity and because of the lack of natural boundary conditions, we work on the torus, i.e. the spatial variable $x \in \mathbb{T} = [0, 2\pi]^3$ and periodic boundary conditions are assumed. We set $L_p = L^p(\mathbb{T})$.

As usual in the periodic setting, we can restrict ourselves to deal with initial data with vanishing spatial averages; then the solutions will enjoy the same property at any fixed time $t > 0$.

Therefore we can represent any $\mathbb{T}$-periodic function $f : \mathbb{R}^3 \to \mathbb{R}$ as

$$f(x) = \sum_{k \in \mathbb{Z}^3_0} f_k e^{ik \cdot x}, \qquad \text{with } f_k \in \mathbb{C},\ f_{-k} = \overline{f}_k \quad \forall k$$

where $\mathbb{Z}_0^3 = \mathbb{Z}^3 \setminus 0$. For $s \in \mathbb{R}$ we define the spaces

$$H^s = \{f = \sum_{k \in \mathbb{Z}_0^3} f_k e^{ik\cdot x} : f_{-k} = \overline{f}_k \text{ and } \sum_{k \in \mathbb{Z}_0^3} |f_k|^2 |k|^{2s} < \infty\}.$$

They are a Hilbert spaces with scalar product

$$\langle f, g\rangle_{H^s} = \sum_{k \in \mathbb{Z}_0^3} f_k g_{-k} |k|^{2s}.$$

We simply denote by $\langle f, g\rangle$ the scalar product in H^0 and also the dual pairing of $H^s - H^{-s}$, i.e. $\langle f, g\rangle = \sum_k f_k g_{-k}$.

The space $H^{s+\epsilon}$ is compactly embedded in H^s for any $\epsilon > 0$. Moreover, we recall the Sobolev embeddings: if $0 \le s < \frac{3}{2}$ and $\frac{1}{p} = \frac{1}{2} - \frac{s}{3}$, then $H^s \subset L_p$ and there exists a constant C (depending on s and p) such that

$$\|f\|_{L_p} \le C\|f\|_{H^s}. \tag{5}$$

If $s = \frac{3}{2}$, then

$$\|f\|_{L_p} \le C\|f\|_{H^s} \qquad \text{for any finite } p$$

and if $s > \frac{3}{2}$, then

$$\|f\|_{L_\infty} \le C\|f\|_{H^s}.$$

We shall often use the following inequality, merging the two latter ones:

$$\text{if } s \ge \frac{3}{2} \text{ then } \|f\|_{L_p} \le C\|f\|_{H^s} \qquad \text{for any finite } p \tag{6}$$

Hereafter, we denote by the same symbol C different constants.

Similarly, we define the spaces for the divergence free velocity vectors, which are periodic and have zero spatial average. For $w : \mathbb{R}^3 \to \mathbb{R}^3$ we write formally

$$w(x) = \sum_{k \in \mathbb{Z}_0^3} w_k e^{ik\cdot x}, \qquad \text{with } w_k \in \mathbb{C}^3,\ w_{-k} = \overline{w}_k, w_k \cdot k = 0 \quad \forall k$$

and for $s \in \mathbb{R}$ define

$$V^s = \{w = \sum_{k \in \mathbb{Z}_0^3} w_k e^{ik\cdot x} : w_{-k} = \overline{w}_k,\ w_k \cdot k = 0 \text{ and } \sum_{k \in \mathbb{Z}_0^3} |w_k|^2 |k|^{2s} < \infty\}.$$

This is a Hilbert space with scalar product

$$\langle v, w\rangle_{V^s} = \sum_{k \in \mathbb{Z}_0^3} v_k \cdot w_{-k} |k|^{2s}.$$

We define the linear operator $\Lambda = (-\Delta)^{1/2}$, i.e.

$$f = \sum_{k \in \mathbb{Z}_0^3} f_k e^{ik\cdot x} \Longrightarrow \Lambda f = \sum_{k \in \mathbb{Z}_0^3} |k| f_k e^{ik\cdot x}$$

and its powers Λ^s: $\Lambda^s f = \sum_{k\in\mathbb{Z}^3_0} |k|^s f_k e^{ik\cdot x}$; hence $\Lambda^2 = -\Delta$. Note, in particular that Λ^s maps H^r onto H^{r-s}.
For simplicity, we shall use the same notation for Λ in the scalar spaces H^s and in the vector spaces V^s.

Let us denote by Π the Leray-Helmholtz projection from L_2 onto V^0. The operators Π and Λ^s commute.

Finally we define the bilinear operator $B : V^1 \times V^1 \to V^{-1}$ by

$$\langle B(u,v), w\rangle = \int_{\mathbb{T}} \Big((u\cdot\nabla)v\Big)\cdot w \; dx$$

i.e. $B(u,v) = \Pi\Big((u\cdot\nabla)v\Big)$ for smooth vectors u, v.

We summarize the properties of the nonlinear terms; these are classical results, see e.g. [25].

Lemma 2. *For any $u, v, w \in V^1$ and $\theta, \eta \in H^1$ we have*

$$\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle, \quad \langle B(u,v), v\rangle = 0, \tag{7}$$

$$\langle v\cdot\nabla\theta, \eta\rangle = -\langle v\cdot\nabla\eta, \theta\rangle, \qquad \langle v\cdot\nabla\theta, \theta\rangle = 0 \tag{8}$$

(7) *holds more generally for any u, v, w giving a meaning to the trilinear forms, as stated precisely in the following:*

$$\langle B(u,v), w\rangle \le C\|u\|_{V^{m_1}}\|v\|_{V^{1+m_2}}\|w\|_{V^{m_3}} \tag{9}$$

with the non negative parameters fulfilling

$$m_1 + m_2 + m_3 \ge \frac{3}{2} \qquad \text{if } m_i \ne \frac{3}{2} \text{ for any } i$$

or

$$m_1 + m_2 + m_3 > \frac{3}{2} \qquad \text{if } \exists\ m_i = \frac{3}{2}.$$

Now, we are ready to give the abstract formulation of problem (4); we apply the projection operator Π to the first equation in order to get rid of the pressure. In addition due to the periodic setting, we regularize u in a different, but equivalent way. Therefore, our system in abstract form is

$$\begin{cases} \partial_t v + B(u,v) + \nu\Lambda^{2\beta} v = \Pi(\theta e_3) \\ \partial_t\theta + u\cdot\nabla\theta = 0 \\ v = \Lambda^{2\alpha} u \end{cases} \tag{10}$$

We focus our analysis on the unknowns v and θ. The pressure p will be recovered by taking the curl of the equation for the velocity in (4), i.e. p solves the equation $\Delta p = -\nabla\cdot[(u\cdot\nabla)v - \theta e_3] = -\nabla\cdot[((\Lambda^{-2\alpha}v)\cdot\nabla)v - \theta e_3]$. Moreover, the unknown u is directly related to v.

Therefore we give the following definition in terms of v and θ only. The finite time interval $[0,T]$ is fixed throughout the paper.

Definition 3. *Let $\alpha \geq 0$ and $\beta > 0$. We are given $v_0 \in V^0, \theta_0 \in H^0$. We say that the couple (v, θ) is a weak solution to system* (10) *over the time interval $[0, T]$ if*

$$v \in L^\infty(0,T;V^0) \cap L^2(0,T;V^\beta) \cap C_w([0,T];V^0)$$
$$\theta \in L^\infty(0,T;H^0) \cap C_w([0,T];H^0)$$

and, given any $\psi \in V^{\frac{5}{2}}$, $\phi \in H^{\frac{5}{2}}$, they satisfy

$$\begin{aligned}\langle v(t), \psi\rangle - \int_0^t \langle B(u(s),\psi), v(s)\rangle ds + \nu \int_0^t \langle \Lambda^\beta v(s), \Lambda^\beta \psi\rangle ds \\ = \langle v_0, \psi\rangle + \int_0^t \langle \theta(s) e_3, \psi\rangle ds\end{aligned} \tag{11}$$

$$\langle \theta(t), \phi\rangle - \int_0^t \langle u(s)\cdot \nabla\phi, \theta(s)\rangle ds = \langle \theta_0, \phi\rangle \tag{12}$$

for every $t \in [0, T]$.

Remark 2. In the weak formulations above, the trilinear terms are well defined; indeed, if $0 < 2\alpha + \beta \leq \frac{3}{2}$

$$\begin{aligned}|\langle B(u,\psi), v\rangle| &\leq C\|u\|_{V^{2\alpha}} \|\psi\|_{V^{\frac{5}{2}-2\alpha-\beta}} \|v\|_{V^\beta} \qquad \text{by (9)} \\ &\leq C\|v\|_{V^0}\|v\|_{V^\beta}\|\psi\|_{V^{\frac{5}{2}}}\end{aligned} \tag{13}$$

and if $2\alpha + \beta > \frac{3}{2}$

$$\begin{aligned}|\langle B(u,\psi), v\rangle| &\leq C\|u\|_{L_\infty}\|\nabla\psi\|_{L_2}\|v\|_{L_2} && \text{by Hölder inequality} \\ &\leq C\|u\|_{V^{2\alpha+\beta}}\|v\|_{V^0}\|\psi\|_{V^1} && \text{since } V^{2\alpha+\beta} \subset L_\infty \\ &\leq C\|v\|_{V^\beta}\|v\|_{V^0}\|\psi\|_{V^1}.\end{aligned} \tag{14}$$

Similarly for the temperature:
if $0 < 2\alpha + \beta < \frac{3}{2}$

$$\begin{aligned}|\langle u\cdot\nabla\phi, \theta\rangle| &\leq \|u\|_{L_{p_1}}\|\nabla\phi\|_{L_{p_2}}\|\theta\|_{L_2} \\ &\leq C\|u\|_{V^{2\alpha+\beta}}\|\nabla\phi\|_{H^{\frac{3}{2}-2\alpha-\beta}}\|\theta\|_{H^0} \\ &\leq C\|v\|_{V^\beta}\|\phi\|_{H^{\frac{5}{2}}}\|\theta\|_{H^0}\end{aligned} \tag{15}$$

where we used first the Hölder inequality with $\frac{1}{p_1} = \frac{1}{2} - \frac{2\alpha+\beta}{3} \in (0, \frac{1}{2}), \frac{1}{p_2} = \frac{1}{2} - \frac{1}{p_1}$ and then the embedding theorems;
if $2\alpha + \beta \geq \frac{3}{2}$

$$\begin{aligned}|\langle u\cdot\nabla\phi, \theta\rangle| &\leq \|u\|_{L_4}\|\nabla\phi\|_{L_4}\|\theta\|_{L_2} \\ &\leq C\|u\|_{V^{2\alpha+\beta}}\|\nabla\phi\|_{H^{\frac{3}{4}}}\|\theta\|_{H^0} \\ &\leq C\|v\|_{V^\beta}\|\phi\|_{H^{\frac{7}{4}}}\|\theta\|_{H^0} \\ &\leq C\|v\|_{V^\beta}\|\theta\|_{H^0}\|\phi\|_{H^{\frac{5}{2}}}\end{aligned} \tag{16}$$

where we used first the Hölder inequality and then the embedding theorems $V^{2\alpha+\beta} \subset L_q$ for any finite q, $H^{\frac{5}{2}} \subset H^{\frac{7}{4}}$, $H^{\frac{3}{4}} \subset L_4$.

For more regular solutions, the trilinear term $\langle B(u,\psi), v\rangle$ is equal to $-\langle B(u,v),\psi\rangle$ and we recover the term appearing in the equation for the velocity. The same holds for the temperature.

Remark 3. We point out that the estimates by means of Sobolev embeddings need some restriction for the parameters; but, for bigger values of the parameters they are easier to prove and the details will be skipped. This means for instance that (13) with (7) gives

$$\|B(u,v)\|_{V^{-\frac{5}{2}}} \leq C\|v\|_{V^0}\|v\|_{V^\beta}$$

assuming $2\alpha+\beta \leq \frac{3}{2}$, whereas for $2\alpha+\beta > \frac{3}{2}$ we get something stronger in (14):

$$\|B(u,v)\|_{V^{-1}} \leq C\|v\|_{V^0}\|v\|_{V^\beta}$$

which is proven in another way. But for sure, from the proof of (13) one can say that $\|B(u,v)\|_{V^{-\frac{5}{2}}} \leq C\|v\|_{V^0}\|v\|_{V^\beta}$ also for $2\alpha+\beta > \frac{3}{2}$ without proving it.

In this last part of the section, we summarize the technical tools to be used later on.

To estimate an L_∞-norm we use either the embedding theorem $H^r \subset L_\infty$ with $r > \frac{3}{2}$ or the **Brézis-Gallouet-Wainger inequality** (see [5, 6]): for any $r > \frac{3}{2}$ there exists a constant C such that

$$\|g\|_{L_\infty} \leq C\|g\|_{H^{\frac{3}{2}}}\left(1+\sqrt{\ln(1+\frac{\|g\|_{H^r}}{\|g\|_{H^{\frac{3}{2}}}})}\right). \tag{17}$$

Actually, we shall use the stronger form of this inequality, as given for instance in [27]: for any $r > \frac{3}{2}$ there exists a constant C such that

$$\|g\|_{L_\infty} \leq C\left(1+\|g\|_{H^{\frac{3}{2}}}+\|g\|_{H^{\frac{3}{2}}}\ln(e+\|g\|_{H^r})\right). \tag{18}$$

Gagliardo-Niremberg inequality (see [19])
Let $1 \leq q, r \leq \infty$, $0 < s < m$, $\frac{s}{m} \leq a < 1$ and

$$\frac{1}{p} = \frac{s}{3} + \left(\frac{1}{q} - \frac{m}{3}\right)a + \frac{1-a}{r}$$

then there exists a constant C such that

$$\|\Lambda^s g\|_{L_p} \leq C\|g\|_{L_r}^{1-a}\|\Lambda^m g\|_{L_q}^a. \tag{19}$$

Define the commutator

$$[\Lambda^s, f]g = \Lambda^s(fg) - f\ \Lambda^s g.$$

From [14], [16] we have

Lemma 4 (Commutator lemma). *Let $s>0$, $1<p<\infty$ and $p_2,p_3\in(1,\infty)$ be such that*

$$\frac{1}{p}\geq\frac{1}{p_1}+\frac{1}{p_2},\qquad \frac{1}{p}\geq\frac{1}{p_3}+\frac{1}{p_4}.$$

Then

$$\|[\Lambda^s,f]g\|_{L_p}\leq C\left(\|\nabla f\|_{L_{p_1}}\|\Lambda^{s-1}g\|_{L_{p_2}}+\|\Lambda^s f\|_{L_{p_3}}\|g\|_{L_{p_4}}\right).$$

and

Lemma 5. *Let $s>0$, $1<p<\infty$ and $p_2,p_3\in(1,\infty)$ be such that*

$$\frac{1}{p}\geq\frac{1}{p_1}+\frac{1}{p_2},\qquad \frac{1}{p}\geq\frac{1}{p_3}+\frac{1}{p_4}.$$

Then

$$\|\Lambda^s(fg)\|_{L_p}\leq C\left(\|f\|_{L_{p_1}}\|\Lambda^s g\|_{L_{p_2}}+\|\Lambda^s f\|_{L_{p_3}}\|g\|_{L_{p_4}}\right).$$

We shall use the commutator acting also on vectors; in particular for $u,v\in\mathbb{R}^3,\theta\in\mathbb{R}$

$$[\Lambda^s,u]\cdot\nabla\theta=\Lambda^s(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^s\theta$$

and

$$[\Lambda^s,u]\cdot\nabla v=\Lambda^s\big((u\cdot\nabla)v\big)-(u\cdot\nabla)\Lambda^s v.$$

Therefore

$$\langle\Lambda^s(u\cdot\nabla\theta),\Lambda^s\theta\rangle=\langle[\Lambda^s,u]\cdot\nabla\theta,\Lambda^s\theta\rangle+\underbrace{\langle u\cdot\nabla\Lambda^s\theta,\Lambda^s\theta\rangle}_{=0\text{ by (8)}}\tag{20}$$

and

$$\langle\Lambda^s\big((u\cdot\nabla)v\big),\Lambda^s v\rangle=\langle[\Lambda^s,u]\cdot\nabla v,\Lambda^s v\rangle+\underbrace{\langle(u\cdot\nabla)\Lambda^s v,\Lambda^s v\rangle}_{=0\text{ by (7)}}\tag{21}$$

About the continuity in time, we have the strong continuity result (see [23] or Lemma 1.4, Chap III in [24])

Lemma 6. *Let $s\in\mathbb{R}$ and $h>0$.*
If $v\in L^2(0,T;V^{s+h})$ and $\frac{dv}{dt}\in L^2(0,T;V^{s-h})$, then $v\in C([0,T];V^s)$ and

$$\frac{d}{dt}\|v(t)\|^2_{V^s}=2\langle\Lambda^{-h}\frac{dv}{dt}(t),\Lambda^h v(t)\rangle_{V_s}$$

and the weak continuity result (see [23]).

Lemma 7. *Let X and Y be Banach spaces, X reflexive, X a dense subset of Y and the inclusion map of X into Y continuous. Then*

$$L^\infty(0,T;X)\cap C_w([0,T];Y)=C_w([0,T];X).$$

3. Existence of weak solutions

Existence of a global weak solution of system (10) can be obtained easily; the technique is very similar to that for the classical Boussinesq system. The equation for θ is a pure transport equation; then the L_q-norm of θ is conserved in time (for any $q \le +\infty$). On the other hand, it is enough to have some regularization in the velocity equation (i.e. $\beta > 0$) in order to get a weak solution as in Definition 3; moreover, this solution satisfies an energy inequality. Of course, the bigger are the parameters α, β, the more regular is the velocity v.

Theorem 8. *Let $\alpha \ge 0, \beta > 0$ and $2 \le q \le \infty$. For any $v_0 \in V^0, \theta_0 \in L_q$, there exists a weak solution (v, θ) of (10) on the time interval $[0,T]$. Moreover*

$$\theta \in C_w(0,T;L_q).$$

PROOF. We define the finite dimensional projector operator Π_n in V^0 as $\Pi_n v = \sum_{0<|k|\le n} v_k e^{ik\cdot x}$ for $v = \sum_{k\in\mathbb{Z}_0^3} v_k e^{ik\cdot x}$; similarly for the scalar case, i.e. Π_n in H^0. We set $B_n(u,v) = \Pi_n B(u,v)$.

We consider the finite dimensional approximation of system (10) in the unknowns $v_n = \Pi_n v$, $u_n = \Pi_n u$ and $\theta_n = \Pi_n \theta$. This is the Galerkin approximation for $n = 1, 2, \dots$

$$\begin{cases} \partial_t v_n + B_n(u_n, v_n) + \nu \Lambda^{2\beta} v_n = \Pi(\theta_n e_3) \\ \partial_t \theta_n + \Pi_n(u_n \cdot \nabla \theta_n) = 0 \\ v_n = \Lambda^{2\alpha} u_n \end{cases} \tag{22}$$

We take the L_2-scalar product of the equation for the velocity v_n with v_n itself; bearing in mind (7) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_n(t)\|_{V^0}^2 + \nu\|v_n(t)\|_{V^\beta}^2 &= -\langle B_n(u_n(t), v_n(t)), v_n(t)\rangle + \langle \Pi(\theta_n(t)e_3), v_n(t)\rangle \\ &= -\langle B(u_n(t), v_n(t)), v_n(t)\rangle + \langle \theta_n(t)e_3, v_n(t)\rangle \\ &\le \frac{1}{2}\|\theta_n(t)\|_{H^0}^2 + \frac{1}{2}\|v_n(t)\|_{V^0}^2 \end{aligned}$$

and similarly for the second equation

$$\begin{aligned} \frac{d}{dt}\|\theta_n(t)\|_{H^0}^2 &= -\langle \Pi_n(u_n(t)\cdot\nabla\theta_n(t)), \theta_n(t)\rangle \\ &= -\langle u_n(t)\cdot\nabla\theta_n(t), \theta_n(t)\rangle = 0. \end{aligned}$$

In both cases the trilinear forms vanish according to (7), (8).

Adding these estimates, by means of Gronwall's lemma we get the basic L_2-energy estimate: there exists a constant K_1 independent of n such that

$$\sup_{0\le t\le T}(\|v_n(t)\|_{V^0}^2 + \|\theta_n(t)\|_{H^0}^2) + \nu\int_0^T \|v_n(t)\|_{V^\beta}^2 dt \le K_1$$

for any n.

From the equation for the velocity v_n, one has that $\frac{dv_n}{dt}$ is expressed as the sum of three terms involving v_n, u_n and θ_n. In particular, the dissipative term $\Lambda^{2\beta}v_n \in L^2(0,T;V^{-\beta})$; by (13), (14) we have $B_n(u_n,v_n) \in L^2(0,T;V^{-s})$ for some finite $s \geq 1$. Therefore there exist constants $\gamma > 0$ and K_2 independent of n, such that

$$\|\frac{dv_n}{dt}\|^2_{L^2(0,T;V^{-\gamma})} \leq K_2.$$

This means that v_n is bounded in $L^2(0,T;V^\beta)\cap W^{1,2}(0,T;V^{-\gamma})$ (with $\beta > 0$ and $\gamma > 0$), which is compactly embedded in $L^2(0,T;V^0)$ (see Lemma 2.2. in [24]). Hence we can extract a subsequence, still denoted by $\{v_n\}$ and $\{\theta_n\}$, such that

$$\begin{aligned} v_n &\longrightarrow v \quad \text{weakly in} \quad L^2(0,T;V^\beta) \\ v_n &\longrightarrow v \quad \text{weakly}^* \text{ in} \quad L^\infty(0,T;V^0) \\ v_n &\longrightarrow v \quad \text{strongly in} \quad L^2(0,T;V^0) \\ \theta_n &\longrightarrow \theta \quad \text{weakly}^* \text{ in} \quad L^\infty(0,T;H^0). \end{aligned}$$

Using these convergences, it is a classical result to pass to the limit in the variational formulation (11) and (12) and prove that (v,θ) is solution of (10) and inherits all the regularity from (v_n,θ_n), i.e.

$$v \in L^\infty(0,T;V^0) \cap L^2(0,T;V^\beta), \qquad \theta \in L^\infty(0,T;H^0).$$

Moreover, it is a classical result (see [27]) that

$$\sup_{0\leq t\leq T} \|\theta_n(t)\|_{L_q} \leq \|\theta_0\|_{L_q} \tag{23}$$

for any $q \leq \infty$.

Hence, the sequence $\{\theta_n\}_n$ is uniformly bounded in $L^\infty(0,T;L_q)$ which implies (up to a subsequence still denoted θ_n) that

$$\theta_n \longrightarrow \theta \quad \text{weakly}^* \text{ in} \quad L^\infty(0,T;L_q)$$

and

$$\sup_{0\leq t\leq T} \|\theta(t)\|_{L_q} \leq \|\theta_0\|_{L_q}. \tag{24}$$

Now, let us prove that $v \in C_w([0,T];V^0)$ and $\theta \in C_w([0,T];L_q)$.
We integrate in time the equation for v:

$$v(t) = v_0 + \int_0^t [-B(u(s),v(s)) - \nu\Lambda^{2\beta}v(s) + \Pi\theta(s)e_3]ds.$$

Bearing in mind (7) and the estimates of Remark 2, we get that $B(u,v) \in L^2(0,T;V^{-\frac{5}{2}})$; therefore $v \in C([0,T];V^{-m})$ for some positive m. By Lemma 7 we get that $v \in C_w([0,T];V^0)$.

Now we look for the weak continuity of θ. Assume that $\phi \in C^\infty_\#(\mathbb{T})$ which is the space of C^∞ functions on $\mathbb{T}$ that are periodic. Then for $t, s \in [0,T]$, we have that

$$\begin{aligned}
|\langle \theta(t) - \theta(s), \phi\rangle| &= |\int_s^t \langle u(r)\cdot\nabla\phi, \theta(r)\rangle dr| \\
&\leq \int_s^t \|\nabla\phi\|_{L_\infty}\|u(r)\|_{L_2}\|\theta(r)\|_{L_2} dr \\
&\leq \|\nabla\phi\|_{L_\infty}\|\theta\|_{L^\infty(0,T;H^0)}\int_s^t \|u(r)\|_{V^0} dr.
\end{aligned}$$

Using the density of $C^\infty_\#(\mathbb{T})$ in $L_{q'}$ (with $\frac{1}{q} + \frac{1}{q'} \leq 1$), we deduce that

$$\lim_{t\to s}\langle \theta(t) - \theta(s), \phi\rangle = 0 \quad \forall \phi \in L_{q'}$$

which means that $\theta \in C_w([0,T]; L_q)$. A similar argument can be used for $q = \infty$ and this completes the proof. $\square$

Remark 4. Take $\alpha \geq 0$ and $\beta > 0$ such that

$$2\alpha + \beta \leq \frac{3}{2}, \qquad \alpha + \beta \geq \frac{5}{4}.$$

For this to hold it is necessary that α is not too big ($\alpha \leq \frac{1}{4}$) and β not too small ($1 \leq \beta \leq \frac{3}{2}$). Then, from the first estimate in (13) we get $B(u,v) \in L^2(0,T;V^{-\beta})$. Hence, going back to the proof of the previous theorem we get that $\frac{dv}{dt} \in L^2(0,T;V^{-\beta})$; by Lemma 6 this implies that $v \in C([0,T];V^0)$, which is stronger than the weak continuity result of Theorem 8 (see Definition 3).

In addition, for more regular initial data we have

Theorem 9 (More regularity). *We are given parameters α and β with $\frac{1}{2} < \beta < \frac{5}{4}$ and*

$$\alpha + \beta = \frac{5}{4}.$$

Then, given $v_0 \in V^\beta, \theta_0 \in H^0$, any weak solution of (10) *obtained in Theorem 8 is more regular; indeed, the velocity is more regular*

$$v \in C([0,T];V^\beta) \cap L^2(0,T;V^{2\beta}).$$

Proof. We look for a priori estimates for v. We proceed as before, but for more regular norms. We have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{V^\beta} + \nu\|v(t)\|^2_{V^{2\beta}} \\
&\qquad = -\langle B(u(t),v(t)), \Lambda^{2\beta}v(t)\rangle + \langle \Pi\theta(t)e_3, \Lambda^{2\beta}v(t)\rangle \\
&\qquad \leq |\langle \Lambda^\beta\big((u(t)\cdot\nabla)v(t)\big), \Lambda^\beta v(t)\rangle| + \|\theta(t)\|_{L_2}\|\Lambda^{2\beta}v(t)\|_{L_2} \\
&\qquad \leq \|[\Lambda^\beta, u(t)]\cdot\nabla v(t)\|_{L_2}\|\Lambda^\beta v(t)\|_{L_2} + \|\theta(t)\|_{H^0}\|v(t)\|_{V^{2\beta}}
\end{aligned}$$

where we used (21).

We use the Commutator Lemma 4

$$
\begin{aligned}
\|[\Lambda^\beta, u]\cdot\nabla v\|_{L_2} &\le C\left(\|\Lambda u\|_{L_{P_1}}\|\Lambda^\beta v\|_{L_{p_2}} + \|\Lambda^\beta u\|_{L_{p_3}}\|\nabla v\|_{L_{p_4}}\right)\\
&= C\left(\|\Lambda^{1-2\alpha} v\|_{L_{p_1}}\|\Lambda^\beta v\|_{L_{p_2}} + \|\Lambda^{\beta-2\alpha} v\|_{L_{p_3}}\|\nabla v\|_{L_{p_4}}\right)
\end{aligned}
\tag{25}
$$

and we want to estimate further with $C\|v\|_{V^\beta}\|v\|_{V^{2\beta}}$.

For this we take

$$
\frac{1}{p_1} = \frac{1}{2} - \frac{\beta - 1 + 2\alpha}{3} \equiv \frac{\beta}{3}, \qquad\qquad \frac{1}{p_2} = \frac{1}{2} - \frac{\beta}{3}.
$$

The assumption $\frac{1}{2} < \beta < \frac{5}{4}$ provides $3 < p_2 < 12$ and by Sobolev embedding

$$
\|\Lambda^{1-2\alpha} v\|_{L_{p_1}} \le C\|v\|_{V^\beta}, \qquad \|\Lambda^\beta v\|_{L_{p_2}} \le C\|v\|_{V^{2\beta}}.
$$

For the latter two terms in (25) we choose

$$
\frac{1}{p_3} = \frac{1}{2} - \frac{2\alpha}{3}, \qquad \frac{1}{p_4} = \frac{1}{2} - \frac{1}{p_3} \equiv \frac{1}{2} - \frac{2\beta - 1}{3}.
$$

The assumption $0 < \alpha < \frac{3}{4}$, i.e. $\frac{1}{2} < \beta < \frac{5}{4}$, provides $2 < p_3 < \infty$ and by Sobolev embedding

$$
\|\Lambda^{\beta-2\alpha} v\|_{L_{p_3}} \le C\|v\|_{V^\beta}, \qquad \|\nabla v\|_{L_{p_4}} \le C\|v\|_{V^{2\beta}}.
$$

Hence, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{V^\beta} + \nu\|v(t)\|^2_{V^{2\beta}}\\
&\quad\le C\|v(t)\|_{V^\beta}\|v(t)\|_{V^{2\beta}}\|v(t)\|_{V^\beta} + C\|\theta(t)\|_{H^0}\|v(t)\|_{V^{2\beta}}\\
&\quad\le \frac{\nu}{2}\|v(t)\|^2_{V^{2\beta}} + C_\nu\|v(t)\|^4_{V^\beta} + C_\nu\|\theta(t)\|^2_{H^0}
\end{aligned}
\tag{26}
$$

by Young inequality. In particular,

$$
\frac{d}{dt}\|v(t)\|^2_{V^\beta} \le C_\nu\|v\|^4_{V^\beta} + C_\nu\|\theta\|^2_{H^0}.
$$

Since $v \in L^2(0,T;V^\beta)$ and $\theta \in L^\infty(0,T;H^0)$ from the previous theorem, we can proceed by means of Gronwall lemma to get the estimate for the $L^\infty(0,T;V^\beta)$-norm:

$$
\sup_{0\le t\le T}\|v(t)\|^2_{V^\beta} \le \|v_0\|^2_{V^\beta} e^{C_\nu\int_0^T\|v(s)\|^2_{V^\beta}ds} + C_\nu\int_0^T e^{C_\nu\int_r^T\|v(s)\|^2_{V^\beta}ds}\|\theta(r)\|^2_{H^0}dr.
$$

Integrating in time (26), we also get

$$
\frac{\nu}{2}\int_0^T\|v(t)\|^2_{V^{2\beta}}\,dt \le \frac{1}{2}\|v_0\|^2_{V^\beta} + C_\nu\|v\|^4_{L^\infty(0,T;V^\beta)} + C_\nu\int_0^T\|\theta(t)\|^2_{H^0}dt.
$$

Summing up, we get that $v \in L^\infty(0,T;V^\beta) \cap L^2(0,T;V^{2\beta})$.

Now, we study the time regularity. We recall property (9) for the nonlinear term $B(u,v)$ with $m_1 = 2\alpha > 0$, $m_2 = 2\beta - 1 > 0$, $m_3 = 0$ (we are in the first case, with all $m_i \neq \frac{3}{2}$ and thus we take $m_1 + m_2 + m_3 = \frac{3}{2}$). We have

$$\begin{aligned}\|\frac{dv}{dt}(t)\|_{L_2} &\leq \|B(u(t),v(t))\|_{L_2} + \nu\|\Lambda^{2\beta}v(t)\|_{L_2} + \|\theta(t)e_3\|_{L_2}\\ &\leq C\|u(t)\|_{V^{2\alpha}}\|v(t)\|_{V^{2\beta}} + \nu\|v(t)\|_{V^{2\beta}} + \|\theta(t)\|_{H^0}\\ &= C\|v(t)\|_{V^0}\|v(t)\|_{V^{2\beta}} + \nu\|v(t)\|_{V^{2\beta}} + \|\theta(t)\|_{H^0}\end{aligned}$$

Hence, using the regularity of v, θ we get that

$$\frac{dv}{dt} \in L^2(0,T;V^0).$$

Now using Lemma 6, we deduce that $v \in C([0,T];V^\beta)$. □

Remark 5. The result of Theorem 9 still holds true under the assumption that $\alpha + \beta > \frac{5}{4}$ with $\beta > \frac{1}{2}$. This is trivial when we add the condition $\alpha > 0$, since the framework is similar to (but easier than) that in the above proof. So, it remains to consider the case $\alpha = 0$ and $\beta > \frac{5}{4}$. To estimate the r.h.s. in (25) we choose $p_1 = \frac{12}{5}, p_2 = 12$, $p_3 = 2$, $p_4 = \infty$ so to get

$$\|\Lambda v\|_{L_{p_1}} \leq C\|v\|_{V^{\frac{5}{4}}} \leq C\|v\|_{V^\beta}$$

$$\|\Lambda^\beta v\|_{L_{p_2}} \leq C\|v\|_{V^{\frac{5}{4}+\beta}} \leq \|v\|_{V^{2\beta}}$$

$$\|\nabla v\|_{L_{p_4}} \leq C\|v\|_{V^{2\beta}}$$

In the study of the time regularity, we choose $m_1 = m_3 = 0$ and $m_2 = 2\beta - 1 > \frac{3}{2}$ and conclude as above.

Similar remarks hold for the proofs of the Appendix, which are still valid when assuming $\alpha + \beta > \frac{5}{4}$ with $\beta > \frac{1}{2}$.

However, our technique requires $\beta > \frac{1}{2}$. This might be improved as in [4]; this is postponed to future work.

4. Regular solutions: global existence, uniqueness and continuous dependence on the initial data

The regularity of solutions from the previous section is not enough to prove uniqueness. To this end, we seek classical solutions. These are solutions for which the spatial derivatives in the equations of (10) exist. Indeed, we shall get that $v \in C([0,T];V^r) \cap L^2(0,T;V^{r+\beta})$ and $\theta \in C([0,T];H^{r-\beta})$ with $r > \beta + 1$. The crucial point is to show that these regular solutions are defined on any given time interval $[0,T]$; their local existence is easy to prove.

Unlike the previous section, here we will consider H^s-regularity for $\theta(t)$ (with $s > 0$). This will help prove the uniqueness of solutions.

Theorem 10. *We are given non negative parameters with* $\frac{1}{2} < \beta < \frac{5}{4}$ *and*

$$\alpha + \beta = \frac{5}{4}. \tag{27}$$

Let

$$r > \max\left(2\beta, \beta + 1\right).$$

Then, for any $v_0 \in V^r, \theta_0 \in H^{r-\beta}$*, there exists a solution* (v, θ) *to* (10) *such that*

$$v \in C([0,T]; V^r) \cap L^2(0,T; V^{r+\beta}), \qquad \theta \in C([0,T]; H^{r-\beta}).$$

Proof. We proceed as before. We take the L_2-scalar product of the first equation of (10) with $\Lambda^{2r}v$; then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{V^r} + \nu\|v(t)\|^2_{V^{r+\beta}} \\
&\quad = -\langle B(u(t), v(t)), \Lambda^{2r}v(t)\rangle + \langle \theta(t)e_3, \Lambda^{2r}v(t)\rangle \\
&\quad = -\langle B(\Lambda^{-2\alpha}v(t), v(t)), \Lambda^{2r}v(t)\rangle + \langle \Lambda^{r-\beta}\theta(t)e_3, \Lambda^{r+\beta}v(t)\rangle \\
&\quad \le C\|v(t)\|_{V^{2\beta}}\|v(t)\|_{V^{r+\beta}}\|v(t)\|_{V^r} + C\|\theta(t)\|_{H^{r-\beta}}\|v(t)\|_{V^{r+\beta}} \\
&\quad \le \frac{\nu}{4}\|v(t)\|^2_{V^{r+\beta}} + C_\nu\|v(t)\|^2_{V^{2\beta}}\|v(t)\|^2_{V^r} + C_\nu\|\theta(t)\|^2_{H^{r-\beta}}
\end{aligned} \tag{28}$$

where we used first Lemma 13 and then Young inequality.

Now for θ, we take the L_2-scalar product of the second equation of (10) with $\Lambda^{2r-2\beta}\theta(t)$; then

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2_{H^{r-\beta}} = -\langle u(t)\cdot\nabla\theta(t), \Lambda^{2r-2\beta}\theta(t)\rangle.$$

We estimate the r.h.s.

$$
\begin{aligned}
\langle u\cdot\nabla\theta,\Lambda^{2r-2\beta}\theta\rangle &= \langle \Lambda^{r-\beta}(u\cdot\nabla\theta),\Lambda^{r-\beta}\theta\rangle \\
&= \langle [\Lambda^{r-\beta},u]\cdot\nabla\theta,\Lambda^{r-\beta}\theta\rangle \quad \text{by (20)} \\
&\le \|[\Lambda^{r-\beta},\Lambda^{-2\alpha}v]\cdot\nabla\theta\|_{L_2}\|\Lambda^{r-\beta}\theta\|_{L_2}
\end{aligned}
$$

and the Commutator Lemma 4 gives

$$
\le C\left(\|\Lambda^{1-2\alpha}v\|_{L_\infty}\|\Lambda^{r-\beta}\theta\|_{L_2}+\|\Lambda^{r-\beta-2\alpha}v\|_{L_{q_3}}\|\Lambda\theta\|_{L_{q_4}}\right)\|\Lambda^{r-\beta}\theta\|_{L_2}
$$

with $\frac{1}{q_3}+\frac{1}{q_4}\le\frac12$; we continue by means of the Brézis-Gallouet-Wainger estimate (18) (with $g=\Lambda^{1-2\alpha}v$) and Lemma 17

$$
\begin{aligned}
&\le C\left(1+\|\Lambda^{\frac52-2\alpha}v\|_{L_2}+\|\Lambda^{\frac52-2\alpha}v\|_{L_2}\ln(e+\|v\|_{V^{m+1-2\alpha}})\right)\|\theta\|^2_{H^{r-\beta}} \\
&\quad + C\|v\|^a_{V^{2\beta}}\|v\|^{1-a}_{V^{r+\beta}}\|\theta\|^{1-a}_{L_q}\|\theta\|^{1+a}_{H^{r-\beta}}
\end{aligned}
$$

for any $m>\frac32$ and for suitable $q>2$, $a\in(0,1)$; m will be chosen later on. Finally we use that $V^{2\beta}=V^{\frac52-2\alpha}$ and $V^{m+1-2\alpha}=V^{m+2\beta-\frac32}$, since $\alpha+\beta=\frac54$:

$$
\begin{aligned}
&\le C\left(1+\|v\|_{V^{2\beta}}+\|v\|_{V^{2\beta}}\ln(e+\|v\|_{V^{m+2\beta-\frac32}})\right)\|\theta\|^2_{H^{r-\beta}} \\
&\quad + C\|v\|^a_{V^{2\beta}}\|v\|^{1-a}_{V^{r+\beta}}\|\theta\|^{1-a}_{L_q}\|\theta\|^{1+a}_{H^{r-\beta}}.
\end{aligned}
$$

Now, we use Young inequality:

$$
\|v\|^a_{V^{2\beta}}\|v\|^{1-a}_{V^{r+\beta}}\|\theta\|^{1-a}_{L_q}\|\theta\|^{1+a}_{H^{r-\beta}}\le\frac{\nu}{4}\|v\|^2_{V^{r+\beta}}+C_\nu\|v\|^{\frac{2a}{1+a}}_{V^{2\beta}}\|\theta\|^{\frac{2(1-a)}{1+a}}_{L_q}\|\theta\|^2_{H^{r-\beta}}.
$$

Set $\phi:=\|v\|^{\frac{2a}{1+a}}_{V^{2\beta}}\|\theta\|^{\frac{2(1-a)}{1+a}}_{L_q}$; then $\phi\in L^1(0,T)$ according to Theorem 9 and (24). Thus

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\theta(t)\|^2_{H^{r-\beta}} &\le C\big(1+\|v(t)\|_{V^{2\beta}}+\|v(t)\|_{V^{2\beta}}\ln(e+\|v(t)\|_{V^{m+2\beta-\frac32}})\big)\|\theta(t)\|^2_{H^{r-\beta}} \\
&\quad +\frac{\nu}{4}\|v(t)\|^2_{V^{r+\beta}}+C_\nu\phi(t)\|\theta(t)\|^2_{H^{r-\beta}}. \qquad (29)
\end{aligned}
$$

Adding the estimates (28) for v and (29) for θ, we get

$$
\begin{aligned}
&\frac{d}{dt}(\|v(t)\|^2_{V^r}+\|\theta(t)\|^2_{H^{r-\beta}})+\nu\|v(t)\|^2_{V^{r+\beta}}\le C\|v(t)\|^2_{V^{2\beta}}\|v(t)\|^2_{V^r} \\
&+C\big(1+\|v(t)\|_{V^{2\beta}}+\|v(t)\|_{V^{2\beta}}\ln(e+\|v\|_{V^{m+2\beta-\frac32}})+\phi(t)\big)\|\theta(t)\|^2_{H^{r-\beta}}. \qquad (30)
\end{aligned}
$$

Recall that $r>2\beta$ by assumption; then there exists $m>\frac32$ such that $V^r\subset$

$V^{m+2\beta-\frac{3}{2}}$. Thus, we get

$$\frac{d}{dt}(\|v(t)\|^2_{V^r}+\|\theta(t)\|^2_{H^{r-\beta}})+\nu\|v(t)\|^2_{V^{r+\beta}}\le C\|v(t)\|^2_{V^{2\beta}}\|v(t)\|^2_{V^r}$$
$$+C\Big(1+\|v(t)\|_{V^{2\beta}}+\|v(t)\|_{V^{2\beta}}\ln(e+\|v(t)\|_{V^r})+\phi(t)\Big)\|\theta(t)\|^2_{H^{r-\beta}} \quad (31)$$

Set $X(t)=\|v(t)\|^2_{V^r}+\|\theta(t)\|^2_{H^{r-\beta}}$. Then, from (31) we easily get

$$\begin{aligned}\frac{dX}{dt}(t) &\le C\Big(1+\|v(t)\|_{V^{2\beta}}\ln(e+1+X(t))+\|v(t)\|^2_{V^{2\beta}}+\phi(t)\Big)X(t)\\ &\le C\Big(1+\|v(t)\|_{V^{2\beta}}\ln(e+1+X(t))+\|v(t)\|^2_{V^{2\beta}}+\phi(t)\Big)(e+1+X(t)).\end{aligned}$$

This implies that $Y(t)=\ln(e+1+X(t))$ satisfies

$$Y'(t)\le C\Big(1+\|v(t)\|_{V^{2\beta}}Y(t)+\|v(t)\|^2_{V^{2\beta}}+\phi(t)\Big).$$

By Gronwall lemma we get

$$\sup_{0\le t\le T}Y(t)\le Y(0)e^{C\int_0^T\|v(s)\|_{V^{2\beta}}ds}+C\int_0^T e^{C\int_s^T\|v(r)\|_{V^{2\beta}}dr}\big(1+\|v(s)\|^2_{V^{2\beta}}+\phi(s)\big)ds.$$

Since $v\in L^2(0,T;V^{2\beta})$ by Theorem 9 and $\phi\in L^1(0,T)$, we get that

$$\sup_{0\le t\le T}Y(t)\le K_3$$

and therefore going back to the unknown X

$$\sup_{0\le t\le T}X(t)\le K_4;$$

from (31), after integration on $[0,T]$ we get also

$$\int_0^T\|v(t)\|^2_{V^{r+\beta}}dt\le K_5.$$

Therefore we have proved that

$$v\in L^\infty(0,T;V^r)\cap L^2(0,T;V^{r+\beta}),\qquad \theta\in L^\infty(0,T;H^{r-\beta}).$$

Now we consider the continuity in time. Lemma 5 (with $p=p_2=2$, $p_1=\infty$) gives

$$\|B(u,v)\|_{V^{r-\beta}}\le C\left(\|u\|_{L_\infty}\|v\|_{V^{r-\beta+1}}+\|\Lambda^{r-\beta}u\|_{L_{p_3}}\|\Lambda v\|_{L_{p_4}}\right).$$

By Sobolev embeddings we get

$$\|u\|_{L_\infty}\le C\|\Lambda^{-2\alpha}v\|_{L_\infty}\le C\|v\|_{V^r}$$

since $r+2\alpha > \frac{3}{2}$ (this comes from the assumption $r > 2\beta = \frac{5}{2} - 2\alpha$), and

$$\|v\|_{V^{r-\beta+1}} \le C\|v\|_{V^{r+\beta}}$$

since $\beta > \frac{1}{2}$.

Now we choose $p_3 \in (2,\infty)$ and p_4 such that $\frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{2}$. When $\beta > 1$ we set $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta+2\alpha}{3} \equiv \frac{\beta-1}{3}$ and $\frac{1}{p_4} = \frac{1}{2} - \frac{\beta-1}{3}$, so to get by Sobolev embedding

$$\|\Lambda^{r-\beta}u\|_{L_{p_3}} = \|\Lambda^{r-\beta-2\alpha}v\|_{L_{p_3}} \le C\|v\|_{V^r} \tag{32}$$

$$\|\Lambda v\|_{L_{p_4}} \le C\|v\|_{V^\beta} \le C\|v\|_{V^{r+\beta}} \qquad \text{for any } r \ge 0$$

whereas when $\beta \le 1$ we have that

$$\|\Lambda v\|_{L_{p_4}} \le C\|\Lambda v\|_{V^{r+\beta-1}} = C\|v\|_{V^{r+\beta}} \tag{33}$$

for some $p_4 \in (2,\infty)$ as soon as $r+\beta-1 > 0$ (take $\frac{1}{p_4} = \frac{1}{2} - \frac{r+\beta-1}{3}$ when $0 < r+\beta-1 < \frac{3}{2}$ and any p_4 finite when $r+\beta-1 \ge \frac{3}{2}$ according to (6)); then in that case we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4} \in (2,\infty)$ and use that (32) holds for any finite p_3 according to (6), since $\beta + 2\alpha = \frac{5}{2} - \beta \ge \frac{3}{2}$.

Hence we have obtained that

$$\|B(u,v)\|_{V^{r-\beta}} \le C\|v\|_{V^r}\|v\|_{V^{r+\beta}}.$$

This implies

$$\frac{dv}{dt} = -B(u,v) - \nu\Lambda^{2\beta}v + \Pi\theta e_3 \in L^2(0,T;V^{r-\beta}).$$

By Lemma 6 we deduce that $v \in C([0,T];V^r)$.

As far as the continuity in time for θ is concerned, we have that θ satisfies a transport equation

$$\partial_t\theta + u\cdot\nabla\theta = 0$$

where the velocity is given and in particular $u \in C([0,T];V^{r+2\alpha})$ with $r+2\alpha > \frac{5}{2}$ (since, by assumption, $r > 2\beta = \frac{5}{2} - 2\alpha$). [13] considers this equation in $\mathbb{R}^2$; but a straightforward modification of Lemma 4.4 of [13] allows to prove in the three dimensional case that given $u \in C([0,T];V^\rho)$ with $\rho > \frac{5}{2}$ and $\theta_0 \in H^k$ with $0 \le k < [\rho]$, then there exists a unique solution $\theta \in C([0,T];H^k)$. Taking $\rho = r+2\alpha$ and $k = r-\beta$, we get the continuity result for θ. $\square$

Now, this regularity is enough to get uniqueness.

Theorem 11 (Uniqueness). *We are given parameters α and β with $\frac{1}{2} < \beta < \frac{5}{4}$ and*

$$\alpha + \beta = \frac{5}{4}.$$

Let

$$r > \max(2\beta, \beta+1).$$

Then, the solutions given in Theorem 10 are unique.

Proof. Let (v_1,θ_1) and (v_2,θ_2) be two solutions given by Theorem 10. We define $V=v_1-v_2$, $U=u_1-u_2$ and $\Phi=\theta_1-\theta_2$. Using the bilinearity we have that they satisfy

$$\begin{cases}\partial_t V+\nu\Lambda^{2\beta}V+B(u_1,V)+B(U,v_2)=\Pi\Phi e_3\\ \partial_t\Phi+U\cdot\nabla\theta_1+u_2\cdot\nabla\Phi=0\end{cases}$$

As before, using (7) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V(t)\|^2_{V^0}+\nu\|V(t)\|^2_{V^\beta}&\\ &=-\langle B(u_1(t),V(t)),V(t)\rangle-\langle B(U(t),v_2(t)),V(t)\rangle+\langle\Phi(t)e_3,V(t)\rangle\\ &\le-\langle B(U(t),v_2(t)),V(t)\rangle+\|\Phi(t)\|_{H^0}\|V(t)\|_{V^0}.\end{aligned}$$

And similarly, using (8)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Phi(t)\|^2_{H^0}&=-\langle(U(t)\cdot\nabla\theta_1(t)),\Phi(t)\rangle-\langle u_2(t)\cdot\nabla\Phi(t),\Phi(t)\rangle\\ &=-\langle(U(t)\cdot\nabla\theta_1(t)),\Phi(t)\rangle.\end{aligned}$$

Let us estimate the terms on the right hand side of each of the relationships above. For the velocity equation, we proceed as usual by means of Hölder and Sobolev inequalities with $\frac{1}{p_2}=\frac{1}{2}-\frac{2\beta-1}{3}\in(0,\frac{1}{2})$ and $\frac{1}{p_1}=\frac{1}{2}-\frac{1}{p_2}\equiv\frac{1}{2}-\frac{2\alpha}{3}$:

$$\begin{aligned}|\langle B(U,v_2),V\rangle|&\le\|(U\cdot\nabla)v_2\|_{L_2}\|V\|_{L_2}\\ &\le\|U\|_{L_{p_1}}\|\nabla v_2\|_{L_{p_2}}\|V\|_{V^0}\\ &\le C\|U\|_{V^{2\alpha}}\|v_2\|_{V^{2\beta}}\|V\|_{V^0}\\ &=C\|V\|_{V^0}\|v_2\|_{V^{2\beta}}\|V\|_{V^0}\\ &\le C\|V\|_{V^\beta}\|v_2\|_{V^{2\beta}}\|V\|_{V^0}\\ &\le\frac{\nu}{4}\|V\|^2_{V^\beta}+C_\nu\|v_2\|^2_{V^{2\beta}}\|V\|^2_{V^0}\end{aligned}$$

Similarly, for the temperature equation:

$$|\langle U\cdot\nabla\theta_1,\Phi\rangle|\le\|U\cdot\nabla\theta_1\|_{L_2}\|\Phi\|_{L_2}\le\|U\|_{L_{p_3}}\|\nabla\theta_1\|_{L_{p_4}}\|\Phi\|_{H^0}$$

with $\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{2}$. Now we choose p_3 and p_4. When $1<\beta<\frac{5}{4}$ we set $\frac{1}{p_3}=\frac{1}{2}-\frac{\frac{5}{2}-\beta}{3}$ and $\frac{1}{p_4}=\frac{1}{2}-\frac{\beta-1}{3}$ so to get $\|U\|_{L_{p_3}}\le C\|U\|_{V^{\frac{5}{2}-\beta}}$ and $\|\nabla\theta_1\|_{L_{p_4}}\le C\|\theta_1\|_{H^\beta}$; in addition we use that $H^{r-\beta}\subseteq H^\beta$ when $r\ge 2\beta$. Therefore

$$\|U\|_{L_{p_3}}\|\nabla\theta_1\|_{L_{p_4}}\le C\|U\|_{V^{\frac{5}{2}-\beta}}\|\theta_1\|_{H^{r-\beta}}=C\|V\|_{V^\beta}\|\theta_1\|_{H^{r-\beta}}. \tag{34}$$

On the other hand, when $\beta\le 1$, according to (6) we have $\|U\|_{L_{p_3}}\le C\|U\|_{V^{\frac{5}{2}-\beta}}$ for any finite p_3; hence we first choose $p_4>2$ such that $\|\nabla\theta_1\|_{L_{p_4}}\le C\|\nabla\theta_1\|_{H^{r-\beta-1}}\le C\|\theta_1\|_{H^{r-\beta}}$; this can be done as soon as $r-\beta-1>0$, i.e. $r>\beta+1$ (as in

(33)). Then we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4}$. Again we have obtained (34).
Thus

$$|\langle U \cdot \nabla \theta_1, \Phi \rangle| \leq C \|V\|_{V^\beta} \|\theta_1\|_{H^{r-\beta}} \|\Phi\|_{H^0} \leq \frac{\nu}{4} \|V\|^2_{V^\beta} + C_\nu \|\theta_1\|^2_{H^{r-\beta}} \|\Phi\|^2_{H^0}.$$

Summing up, we have obtained

$$\begin{aligned} &\frac{d}{dt}\|V(t)\|^2_{V^0} + \nu \|V(t)\|^2_{V^\beta} + \frac{d}{dt}\|\Phi(t)\|^2_{H^0} \\ &\quad \leq C\|v_2(t)\|^2_{V^{2\beta}} \|V(t)\|^2_{V^0} + \|\theta_1(t)\|^2_{H^{r-\beta}} \|\Phi(t)\|^2_{H^0} + \|\Phi(t)\|^2_{H^0} + \|V(t)\|^2_{V^0}. \end{aligned}$$

If we define $Z(t) = \|V(t)\|^2_{V^0} + \|\Phi(t)\|^2_{H^0}$, we have $Z(0) = 0$ and

$$Z'(t) \leq C(\|v_2(t)\|^2_{V^{2\beta}} + \|\theta_1(t)\|^2_{H^{r-\beta}} + 1) Z(t).$$

By Gronwall lemma we get $Z(t) = 0$ for all t, and this completes the proof. □

Theorem 12 (Continuous dependence on the initial data). *We are given parameters α and β with $\frac{1}{2} < \beta < \frac{5}{4}$ and*

$$\alpha + \beta = \frac{5}{4}.$$

Let

$$r > \beta + 2.$$

Then, given any initial conditions $v_{1,0}, v_{2,0} \in V^r$ and $\theta_{1,0}, \theta_{2,0} \in H^{r-\beta}$ we have

$$\begin{aligned} &\|v_1 - v_2\|_{L^\infty(0,T;V^{r-1})} + \|v_1 - v_2\|_{L^2(0,T;V^{r-1+\beta})} + \|\theta_1 - \theta_2\|_{L^\infty(0,T;H^{r-\beta-1})} \\ &\qquad \leq C\left(\|v_{1,0} - v_{2,0}\|_{V^{r-1}} + \|\theta_{1,0} - \theta_{2,0}\|_{H^{r-\beta-1}}\right) \quad (35) \end{aligned}$$

where the constant C depends on T, $\|\theta_1\|_{L^\infty(0,T;H^{r-\beta})}$, $\|v_i\|_{L^2(0,T;V^{r+\beta-1})}$ and $\|v_i\|_{L^\infty(0,T;V^r)}$.

Proof. We begin by pointing out that, under the assumption $\frac{1}{2} < \beta < \frac{5}{4}$ the condition $r > \beta + 2$ implies also $r > \max(2\beta, \beta + 1, 2 - \beta)$ and therefore the assumptions of Theorem 10 and Lemma 14, 15 and 16 are fulfilled.

Using the same setting as in the proof of Theorem 11, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|V(t)\|^2_{V^{r-1}} + \nu\|V(t)\|^2_{V^{r-1+\beta}} &= -\langle B(\Lambda^{-2\alpha} v_1(t), V(t)), \Lambda^{2r-2} V(t)\rangle \\ &\quad - \langle B(\Lambda^{-2\alpha} V(t), v_2(t)), \Lambda^{2r-2} V(t)\rangle + \langle \Lambda^{r-\beta-1}\Phi(t) e_3, \Lambda^{r-1+\beta} V(t)\rangle. \end{aligned}$$

We estimate the first two terms of r.h.s. by means of Lemma 14

$$\begin{aligned} &|\langle B(\Lambda^{-2\alpha} v_1(t), V(t)), \Lambda^{2r-2} V(t)\rangle| + |\langle B(\Lambda^{-2\alpha} V(t), v_2(t)), \Lambda^{2r-2} V(t)\rangle| \\ &\quad \leq C\big(\|v_1\|_{V^r}\|V\|_{V^{r-1}} + \|v_1\|_{V^{r+\beta-1}}\|V\|_{V^{r+\beta-1}}\big)\|V\|_{V^{r-1}} \\ &\qquad + C\|V\|_{V^{r-1}}\|v_2\|_{V^{r+\beta-1}}\|V\|_{V^{r+\beta-1}}. \end{aligned}$$

Using Young inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|V(t)\|^2_{V^{r-1}}+\nu\|V(t)\|^2_{V^{r-1+\beta}}\le\frac{\nu}{2}\|V(t)\|^2_{V^{r+\beta-1}}+C_\nu\|\Phi(t)\|^2_{H^{r-\beta-1}}$$
$$+C_\nu(\|v_1(t)\|_{V^r}+\|v_1(t)\|^2_{V^{r+\beta-1}}+\|v_2(t)\|^2_{V^{r+\beta-1}})\|V(t)\|^2_{V^{r-1}}. \quad (36)$$

Similarly, for the temperature difference; we use Lemma 15 and 16 and Young inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Phi(t)\|^2_{H^{r-\beta-1}}&=-\langle U(t)\cdot\nabla\theta_1(t),\Lambda^{2r-2\beta-2}\Phi(t)\rangle-\langle u_2(t)\cdot\nabla\Phi(t),\Lambda^{2r-2\beta-2}\Phi(t)\rangle\\&\le C\|V(t)\|_{V^{r-1}}\|\theta_1(t)\|_{H^{r-\beta}}\|\Phi(t)\|_{H^{r-\beta-1}}+C\|v_2(t)\|_{V^r}\|\Phi(t)\|^2_{H^{r-\beta-1}}\\&\le C\|V(t)\|^2_{V^{r-1}}+C\big(\|\theta_1(t)\|^2_{H^{r-\beta}}+\|v_2(t)\|_{V^r}\big)\|\Phi(t)\|^2_{H^{r-\beta-1}}.\end{aligned}$$

Finally, we consider the sum $\|V(t)\|^2_{V^{r-1}}+\|\Phi(t)\|^2_{H^{r-\beta-1}}:=W(t)$ and define $a(t)=1+\|\theta_1(t)\|^2_{H^{r-\beta}}+\|v_1(t)\|^2_{V^{r+\beta-1}}+\|v_2(t)\|^2_{V^{r+\beta-1}}+\|v_1(t)\|_{V^r}+\|v_2(t)\|_{V^r}$; we have $a\in L^1(0,T)$ and

$$W'(t)+\nu\|V(t)\|^2_{V^{r+\beta-1}}\le Ca(t)W(t). \quad (37)$$

Gronwall lemma applied to

$$W'(t)\le Ca(t)W(t)$$

gives

$$\sup_{0\le t\le T}W(t)\le W(0)e^{C\int_0^T a(t)\ dt}.$$

Integrating in time (37) and using the latter result we get the estimate for $\int_0^T\|V(t)\|^2_{H^{r+\beta-1}}dt$. This concludes the proof. □

5. Auxiliary results

In this section we prove the lemma used in the proofs of the previous section.

Lemma 13. *Let $\frac{1}{2}<\beta<\frac{5}{4}$ and $\alpha+\beta=\frac{5}{4}$. Then, for any $r>0$ there exists a constant $C>0$ such that*

$$|\langle B(\Lambda^{-2\alpha}v,v),\Lambda^{2r}v\rangle|\le C\|v\|_{V^{2\beta}}\|v\|_{V^{r+\beta}}\|v\|_{V^r}.$$

Proof. Set $u=\Lambda^{-2\alpha}v$. First

$$\begin{aligned}\langle B(u,v),\Lambda^{2r}v\rangle&=\langle\Lambda^r\Big((u\cdot\nabla)v\Big),\Lambda^r v\rangle\\&=\langle[\Lambda^r,u]\cdot\nabla v,\Lambda^r v\rangle\qquad\text{by (21)}\\&\le\|[\Lambda^r,u]\cdot\nabla v\|_{L_2}\|\Lambda^r v\|_{L_2}.\end{aligned}$$

Then, we use the Commutator Lemma 4 with $p=2$ and

$$\tfrac{1}{p_1}=\tfrac{1}{2}-\tfrac{\beta-1+2\alpha}{3},\qquad \tfrac{1}{p_2}=\tfrac{1}{2}-\tfrac{1}{p_1}\equiv\tfrac{1}{2}-\tfrac{\beta}{3}\in(\tfrac{1}{12},\tfrac{1}{3})$$

$$\begin{cases}\tfrac{1}{p_3}=\tfrac{1}{2}-\tfrac{\beta+2\alpha}{3}\in(0,\tfrac{1}{12}),\qquad \tfrac{1}{p_4}=\tfrac{1}{2}-\tfrac{1}{p_3}\equiv\tfrac{1}{2}-\tfrac{\beta-1}{3} & \text{if } 1<\beta<\tfrac{5}{4}\\ \tfrac{1}{p_3}=\tfrac{1}{2}-\tfrac{2\alpha}{3}\in(0,\tfrac{1}{3}],\ \tfrac{1}{p_4}=\tfrac{1}{2}-\tfrac{1}{p_3}\equiv\tfrac{1}{2}-\tfrac{2\beta-1}{3} & \text{if } \tfrac{1}{2}<\beta\leq 1\end{cases}$$

so to get

$$\begin{aligned}\|[\Lambda^r,u]\cdot\nabla v\|_{L_2}&\leq C\left(\|\Lambda u\|_{L_{p_1}}\|\Lambda^r v\|_{L_{p_2}}+\|\Lambda^r u\|_{L_{p_3}}\|\Lambda v\|_{L_{p_4}}\right)\\ &=C\left(\|\Lambda^{1-2\alpha}v\|_{L_{p_1}}\|\Lambda^r v\|_{L_{p_2}}+\|\Lambda^{r-2\alpha}v\|_{L_{p_3}}\|\Lambda v\|_{L_{p_4}}\right).\end{aligned}$$

Then, to conclude our estimate we use the Sobolev embedding inequalities

$$\|\Lambda^{1-2\alpha}v\|_{L_{p_1}}\leq C\|v\|_{V^\beta}\qquad \|\Lambda^r v\|_{L_{p_2}}\leq C\|v\|_{V^{r+\beta}} \tag{38}$$

and for $1<\beta<\frac{5}{4}$

$$\|\Lambda^{r-2\alpha}v\|_{L_{p_3}}\leq C\|v\|_{V^{r+\beta}}\qquad \|\Lambda v\|_{L_{p_4}}\leq C\|v\|_{V^\beta}\leq C\|v\|_{V^{2\beta}},$$

whereas for $\frac{1}{2}<\beta\leq 1$ (i.e. $\frac{1}{4}\leq\alpha<\frac{3}{4}$)

$$\|\Lambda^{r-2\alpha}v\|_{L_{p_3}}\leq C\|v\|_{V^r}\leq C\|v\|_{V^{r+\beta}}\qquad \|\Lambda v\|_{L_{p_4}}\leq C\|v\|_{V^{2\beta}}.$$

□

Lemma 14. *Let $\frac{1}{2}<\beta<\frac{5}{4}$ and $\alpha+\beta=\frac{5}{4}$. If*

$$r>\max(2\beta,2-\beta),$$

then there exists a constant $C>0$ such that

$$|\langle B(\Lambda^{-2\alpha}w,v),\Lambda^{2r-2}v\rangle|\leq C(\|w\|_{V^r}\|v\|_{V^{r-1}}+\|w\|_{V^{r+\beta-1}}\|v\|_{V^{r+\beta-1}})\|v\|_{V^{r-1}}$$

and

$$|\langle B(\Lambda^{-2\alpha}v,w),\Lambda^{2r-2}v\rangle|\leq C\|v\|_{V^{r-1}}\|w\|_{V^{r+\beta-1}}\|v\|_{V^{r+\beta-1}}.$$

Proof. First, notice that we also have $r>1$.
To prove the first inequality, we use the Commutator Lemma 4 with $p=p_2=2$, $p_1=\infty$ and suitables p_3,p_4 to get

$$\begin{aligned}&|\langle B(\Lambda^{-2\alpha}w,v),\Lambda^{2r-2}v\rangle|\\ &\qquad=|\langle\Lambda^{r-1}\Big((\Lambda^{-2\alpha}w\cdot\nabla)v\Big),\Lambda^{r-1}v\rangle|\\ &\qquad=|\langle[\Lambda^{r-1},\Lambda^{-2\alpha}w]\cdot\nabla v,\Lambda^{r-1}v\rangle|\qquad\text{by (21)}\\ &\qquad\leq C(\|\Lambda^{1-2\alpha}w\|_{L_\infty}\|\Lambda^{r-1}v\|_{L_2}+\|\Lambda^{r-1-2\alpha}w\|_{L_{p_3}}\|\Lambda v\|_{L_{p_4}})\|\Lambda^{r-1}v\|_{L_2}\end{aligned}$$

We estimate the first four terms in the latter line. When $1<\beta<\frac{5}{4}$ we choose

$$\tfrac{1}{p_3}=\tfrac{1}{2}-\tfrac{\beta+2\alpha}{3}\in(0,\tfrac{1}{12}),\qquad \tfrac{1}{p_4}=\tfrac{1}{2}-\tfrac{1}{p_3}\equiv\tfrac{1}{2}-\tfrac{\beta-1}{3}, \tag{39}$$

whereas when $\frac{1}{2} < \beta \leq 1$ we choose

$$\begin{cases} \frac{1}{p_4} = \frac{1}{2} - \frac{r+\beta-2}{3} & \text{if } 2 < r+\beta < \frac{7}{2} \\ \text{any } p_4 \in (2,\infty) & \text{if } r+\beta \geq \frac{7}{2} \end{cases} \tag{40}$$

and $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4} \in (0, \frac{1}{2})$.

Then we use the following Sobolev embedding inequalities:

$$\|\Lambda^{1-2\alpha} w\|_{L_\infty} \leq C\|w\|_{V^r}$$

since $r > \frac{5}{2} - 2\alpha = 2\beta$. Moreover, for $1 < \beta < \frac{5}{4}$, according to (39) we have

$$\|\Lambda^{r-1-2\alpha} w\|_{L_{p_3}} \leq C\|w\|_{V^{r+\beta-1}} \qquad \|\Lambda v\|_{L_{p_4}} \leq C\|v\|_{V^\beta}$$

and

$$\|v\|_{V^\beta} \leq C\|v\|_{V^{r-1+\beta}}$$

since $r-1>0$. On the other hand, for $\frac{1}{2} < \beta \leq 1$ according to (40) there exists $p_4 \in (2,\infty)$ such that

$$\|\Lambda v\|_{L_{p_4}} \leq C\|v\|_{V^{r+\beta-1}};$$

then we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4} \in (0, \frac{1}{2})$ and recall that

$$\|\Lambda^{r-1-2\alpha} w\|_{L_{p_3}} = \|\Lambda^{r-\frac{7}{2}+2\beta} w\|_{L_{p_3}} \leq C\|w\|_{V^{r+2\beta-2}} \tag{41}$$

for any finite p_3. Since $\|w\|_{V^{r+2\beta-2}} \leq C\|w\|_{V^{r+\beta-1}}$ when $\beta \leq 1$, this concludes the first inequality of the statement of this Lemma.

For the second inequality, we use Lemma 5 with $p=2$:

$$\begin{aligned} |\langle B(\Lambda^{-2\alpha} v, w), \Lambda^{2r-2} v\rangle| &= |\langle \Lambda^{r-1-\beta}\Big((\Lambda^{-2\alpha} v\cdot\nabla)w\Big), \Lambda^{r+\beta-1} v\rangle| \\ &\leq C\Big(\|\Lambda^{-2\alpha} v\|_{L_{p_1}}\|\Lambda^{r-\beta} w\|_{L_{p_2}} + \|\Lambda^{r-1-\beta-2\alpha} v\|_{L_{p_3}}\|\Lambda w\|_{L_{p_4}}\Big)\|v\|_{V^{r+\beta-1}}. \end{aligned}$$

Now we choose $\frac{1}{p_1} = \frac{1}{2} - \frac{2\alpha}{3} \equiv \frac{2\beta-1}{3}$ and $\frac{1}{p_2} = \frac{1}{2} - \frac{2\beta-1}{3} \in (0, \frac{1}{2})$ since $\frac{1}{2} < \beta < \frac{5}{4}$; then, by means of Sobolev embedding inequalities

$$\|\Lambda^{-2\alpha} v\|_{L_{p_1}} \leq C\|v\|_{V^0} \leq C\|v\|_{V^{r-1}} \text{ for any } r \geq 1$$

$$\|\Lambda^{r-\beta} w\|_{L_{p_2}} \leq C\|w\|_{V^{r+\beta-1}}$$

Moreover, for $1 < \beta < \frac{5}{4}$ we choose $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta+2\alpha}{3} \equiv \frac{\beta-1}{3} \in (0, \frac{1}{12})$ and $\frac{1}{p_4} = \frac{1}{2} - \frac{\beta-1}{3}$; therefore by means of Sobolev embedding theorems we get

$$\|\Lambda^{r-1-\beta-2\alpha} v\|_{L_{p_3}} \leq C\|v\|_{V^{r-1}}$$

$$\|\Lambda w\|_{L_{p_4}} \leq C\|w\|_{V^\beta} \leq C\|w\|_{V^{r-1+\beta}} \text{ for any } r \geq 1$$

On the other side, for $\frac{1}{2} < \beta \le 1$, we choose $p_4 \in (2, \infty)$ as in (40) so to get

$$\|\Lambda w\|_{L_{p_4}} \le C\|w\|_{V^{r-1+\beta}}$$

and we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4}$ to get, as in (41),

$$\|\Lambda^{r-1-\beta-2\alpha} w\|_{L_{p_3}} = \|\Lambda^{r-\frac{7}{2}+\beta} w\|_{L_{p_3}} \le C\|w\|_{V^{r+\beta-2}}.$$

Using that

$$\|v\|_{V^{r+\beta-2}} \le C\|v\|_{V^{r-1}}$$

since $\beta \le 1$, we conclude the second inequality in the statement. □

Lemma 15. *Let $\frac{1}{2} < \beta < \frac{5}{4}$ and $\alpha + \beta = \frac{5}{4}$. If*

$$r > \max(2\beta, \beta + 1)$$

then there exists a constant $C > 0$ such that

$$|\langle \Lambda^{-2\alpha} v \cdot \nabla\theta, \Lambda^{2r-2\beta-2}\phi\rangle| \le C\|v\|_{V^{r-1}}\|\theta\|_{H^{r-\beta}}\|\phi\|_{H^{r-\beta-1}}.$$

Proof. We use Lemma 5 with $p = p_2 = 2$ and $p_1 = \infty$:

$$\begin{aligned}
&|\langle \Lambda^{-2\alpha} v \cdot \nabla\theta, \Lambda^{2r-2\beta-2}\phi\rangle| \\
&\qquad = |\langle \Lambda^{r-\beta-1}(\Lambda^{-2\alpha} v \cdot \nabla\theta), \Lambda^{r-\beta-1}\phi\rangle| \\
&\qquad \le \|\Lambda^{r-\beta-1}(\Lambda^{-2\alpha} v \cdot \nabla\theta)\|_{L_2}\|\phi\|_{H^{r-\beta-1}} \\
&\qquad \le C(\|\Lambda^{-2\alpha} v\|_{L_\infty}\|\Lambda^{r-\beta}\theta\|_{L_2} + \|\Lambda^{r-\beta-1-2\alpha} v\|_{L_{p_3}}\|\Lambda\theta\|_{L_{p_4}})\|\phi\|_{H^{r-\beta-1}}
\end{aligned}$$

We estimate the first four terms in the latter line. Since $r > 2\beta$, i.e. $r - 1 - (2\beta - \frac{5}{2}) > \frac{3}{2}$, we have

$$\|\Lambda^{-2\alpha} v\|_{L_\infty} = \|\Lambda^{2\beta-\frac{5}{2}} v\|_{L_\infty} \le C\|v\|_{V^{r-1}}.$$

For $1 < \beta < \frac{5}{4}$ we set $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta+2\alpha}{3} \in (0, \frac{1}{12})$ and $\frac{1}{p_4} = \frac{1}{2} - \frac{1}{p_3} \equiv \frac{1}{2} - \frac{\beta-1}{3}$ so to get

$$\|\Lambda^{r-\beta-1-2\alpha} v\|_{L_{p_3}} \le C\|v\|_{V^{r-1}}$$

$$\|\Lambda\theta\|_{L_{p_4}} \le C\|\theta\|_{H^\beta} \le C\|\theta\|_{H^{r-\beta}} \quad \text{when } r \ge 2\beta$$

On the other side, when $\frac{1}{2} < \beta \le 1$ we have $\beta + 2\alpha \ge \frac{3}{2}$; hence, according to (6)

$$\|\Lambda^{r-\beta-1-2\alpha} v\|_{L_{p_3}} \le C\|v\|_{V^{r-1}} \qquad \text{for any finite } p_3.$$

Therefore we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4}$ with $p_4 \in (2, \infty)$ chosen arbitrarily when $r \ge \beta + \frac{5}{2}$ and $\frac{1}{p_4} = \frac{1}{2} - \frac{r-\beta-1}{3}$ when $\beta + 1 < r < \beta + \frac{5}{2}$; in this way we get

$$\|\Lambda\theta\|_{L_{p_4}} \le C\|\theta\|_{H^{r-\beta}}.$$

This concludes the proof. □

Lemma 16. *Let $\frac{1}{2} < \beta < \frac{5}{4}$ and $\alpha + \beta = \frac{5}{4}$. If*

$$r > \beta + 2$$

then there exists a constant $C > 0$ such that

$$|\langle \Lambda^{-2\alpha} v \cdot \nabla\theta, \Lambda^{2r-2\beta-2}\theta\rangle| \le C\|v\|_{V^r}\|\theta\|^2_{H^{r-\beta-1}}.$$

PROOF. We use the Commutator Lemma 4 with $p = p_2 = 2$, $p_1 = \infty$:

$$\begin{aligned}
&|\langle \Lambda^{-2\alpha} v \cdot \nabla\theta, \Lambda^{2r-2\beta-2}\theta\rangle| \\
&\quad = |\langle \Lambda^{r-\beta-1}(\Lambda^{-2\alpha} v \cdot \nabla\theta), \Lambda^{r-\beta-1}\theta\rangle| \\
&\quad = |\langle [\Lambda^{r-\beta-1}, \Lambda^{-2\alpha} v] \cdot \nabla\theta, \Lambda^{r-\beta-1}\theta\rangle| \qquad \text{by (20)} \\
&\quad \le C(\|\Lambda^{1-2\alpha} v\|_{L_\infty}\|\Lambda^{r-\beta-1}\theta\|_{L_2} + \|\Lambda^{r-\beta-1-2\alpha} v\|_{L_{p_3}}\|\Lambda\theta\|_{L_{p_4}})\|\theta\|_{H^{r-\beta-1}}
\end{aligned}$$

We estimate the first four terms in the latter line. Since $r > \beta + 2 > 2\beta - 1$ we have

$$\|\Lambda^{1-2\alpha} v\|_{L_\infty} = \|\Lambda^{2\beta-\frac{5}{2}} v\|_{L_\infty} \le C\|v\|_{V^r}.$$

Moreover we have

$$\|\Lambda^{r-\beta-1-2\alpha} v\|_{L_{p_3}} \le C\|\Lambda^{r-\frac{9}{4}} v\|_{L_{p_3}}$$

and according to (6)

$$\|\Lambda^{r-\frac{9}{4}} v\|_{L_{p_3}} \le C\|v\|_{V^r}$$

for any finite p_3. Hence we set $\frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_4}$ with $p_4 \in (2, \infty)$ chosen arbitrarily when $r \ge \beta + \frac{7}{2}$ and $\frac{1}{p_4} = \frac{1}{2} - \frac{r-\beta-2}{3}$ when $\beta + 2 < r < \beta + \frac{7}{2}$ in order to have the Sobolev inequality

$$\|\Lambda\theta\|_{L_{p_4}} \le C\|\theta\|_{H^{r-\beta-1}}.$$

□

Lemma 17. *Let $\alpha + \beta = \frac{5}{4}$ with $\frac{1}{2} < \beta < \frac{5}{4}$ and $r > \beta + 1$. Then, there exist $q_3, q_4 > 2$ with $\frac{1}{q_3} + \frac{1}{q_4} \le \frac{1}{2}$ and $q > 2$, $a \in (0,1)$, $C > 0$ such that*

$$\|\Lambda^{r-\beta-2\alpha} v\|_{L_{q_3}}\|\Lambda\theta\|_{L_{q_4}}\|\theta\|_{H^{r-\beta}} \le C\|v\|^a_{V^{2\beta}}\|v\|^{1-a}_{V^{r+\beta}}\|\theta\|^{1-a}_{L_q}\|\theta\|^{1+a}_{H^{r-\beta}}.$$

PROOF. We use Sobolev embedding theorem, interpolation theorem and the Gagliardo-Nirenberg inequality; then for some $a \in (0,1)$ and $q \ge 2$ to be defined later on we look for

$$\begin{cases}
\|\Lambda^{r-\beta-2\alpha} v\|_{L_{q_3}} \le C\|\Lambda^{r+\beta-ra+\beta a} v\|_{L_2} \le C\|v\|^a_{V^{2\beta}}\|v\|^{1-a}_{V^{r+\beta}} & \text{for } \frac{1}{q_3} = \frac{1}{2} - \frac{2\beta+\beta a-ra+2\alpha}{3} \\
\|\Lambda\theta\|_{L_{q_4}} \le C\|\theta\|^{1-a}_{L_q}\|\Lambda^{r-\beta}\theta\|^a_{L_2} & \text{for } \frac{1}{q_4} = \frac{1}{3} + (\frac{1}{2} - \frac{r-\beta}{3})a + \frac{1-a}{q}
\end{cases}$$

under the conditions

$$\begin{cases}
r + \beta - ra + \beta a \ge r - \beta - 2\alpha \\
\frac{1}{q_3} + \frac{1}{q_4} \le \frac{1}{2} \\
\frac{1}{r-\beta} < a < 1
\end{cases}$$

equivalent to (since $r > \beta$ by assumption)

$$\begin{cases} a \leq \frac{5}{2(r-\beta)} \\ \frac{a}{2} + \frac{1-a}{q} \leq \frac{1}{2} \\ \frac{1}{r-\beta} < a < 1 \end{cases} \tag{42}$$

The second equation is satisfied for some q (big enough) when $0 < a < 1$; therefore we choose $a \in (0,1)$ such that

$$\frac{1}{r-\beta} < a < \min\Big(1, \frac{5}{2(r-\beta)}\Big). \tag{43}$$

This double condition has solutions since $r - \beta > 1$. $\square$

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References

[1] H. Abidi, T. Hmidi: On the global well-posedness for Boussinesq system, *J. Differential Equations* 233 (2007), no. 1, 199-220

[2] H. Abidi, T. Hmidi, S. Keraani: On the global regularity of axisymmetric Navier-Stokes-Boussinesq system, *Discrete Contin. Dyn. Syst.* 29 (2011), no. 3, 737-756

[3] D. Barbato, H. Bessaih, B. Ferrario: On a stochastic Leray-α model of Euler equations, *Stochastic Process. Appl.* **124** (2014), no. 1, 199-219

[4] D. Barbato, F. Morandin, M. Romito: Global regularity for a slightly supercritical hyperdissipative Navier-Stokes system, *Analysis and PDE* **7** (2014), no. 8, 2009-2027

[5] H. Brézis, T. Gallouet: Nonlinear Schrödinger evolution equations, *Nonlinear Anal.* 4 (1980), no. 4, 677-681

[6] H. Brézis, S. Wainger: A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* 5 (1980), no. 7, 773-789

[7] D. Chae: Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Adv. Math.* 203 (2006), no. 2, 497-513

[8] R. Danchin, M. Paicu: Existence and uniqueness results for the Boussinesq system with data in Lorentz spaces, *Phys. D* 237 (2008), no. 10-12, 1444-1460

[9] R. Danchin, M. Paicu: Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data, *Comm. Math. Phys.* 290 (2009), no. 1, 1-14

[10] J. Geng, J. Fan: A note on regularity criterion for the 3D Boussinesq system with zero thermal conductivity, *Appl. Math. Lett.* 25 (2012), no. 1, 63-66

[11] T. Hmidi, Keraani: On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity, *Adv. Differential Equations* 12 (2007), no. 4, 461-480

[12] T. Y. Hou, C. Li: Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Syst.* 12 (2005), no. 1, 1-12

[13] T. Kato, G. Ponce: Well posedness of the Euler and Navier-Stokes equations in the Lebesgues spaces $L^p_s(\mathbb{R}^2)$, *Rev. Mat. Iberoam.* 2 (1986), no. 1-2, 73-88

[14] T. Kato, G. Ponce: Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.* 41 (1988), no. 7, 891-907

[15] M. Kaya, O. Çelebi: Global attractor for the regularized Bénard problem, *Applicable Analysis* 93 (2014), no. 9, 1989-2001

[16] C. Kenig, G. Ponce, L. Vega: Well-posedness of the initial value problem for the Korteweg-de-Vries equation, *J. Amer. Math. Soc.* 4 (1991), 323-347

[17] J.-L. Lions: *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Gauthier-Villars, Paris 1969

[18] J. C. Mattingly, Ya. G. Sinai: An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations, *Commun. Contemp. Math.* 1 (1999), no. 4, 497-516

[19] L. Nirenberg: On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* 13 (1959), no. 2, 115-162

[20] E. Olson, E. S. Titi: Viscosity versus vorticity stretching: global well-posedness for a family of Navier-Stokes-alpha-like models, *Nonlinear Anal.* 66 (2007), no. 11, 2427-2458

[21] H. Qiu, Y. Du, Z. Yao: A blow-up criterion for 3D Boussinesq equations in Besov spaces, *Nonlinear Anal.* 73 (2010), no. 3, 806-815

[22] R. Selmi: Global Well-Posedness and Convergence Results for the 3D-Regularized Boussinesq System, *Canad. J. Math.* 64 (2012), no. 6, 1415-1435

[23] W. A. Strauss: On continuity of functions with values in various Banach spaces, *Pacific J. Math.* 19 (1966), 543-551

[24] R. Temam: *Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications 2*, North-Holland Publishing Co., Amsterdam-New York, 1979.

[25] R. Temam: *Navier-Stokes equations and nonlinear functional analysis*, Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[26] K. Yamazaki: On the global regularity of generalized Leray-alpha type models, *Nonlin. Anal.* 75 (2012), 503–515

[27] Z. Ye: A note on global well-posedness of solutions to Boussinesq equations with fractional dissipation, *Acta Math. Sci.* Ser. B Engl. Ed. 35 (2015), no. 1, 112-120

[28] Xiang Zhaoyin, Yan Wei: Global regularity of solutions to the Boussinesq equations with fractional diffusion, *Adv. Differential Equations* 18 (2013), no. 11-12, 1105-1128